

Sample selection models for discrete and other non-Gaussian response variables

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14th September 2016

Abstract

Consider observation of a phenomenon of interest subject to selective sampling due to a censoring mechanism regulated by some other variable. In this context, an extensive literature exists linked to the so-called Heckman selection model. A great deal of this work has been developed under Gaussian assumption of the underlying probability distributions; considerably less work has dealt with other distributions. We examine a general construction which encompasses a variety of distributions and allows various options of the selection mechanism, focusing especially on the case of discrete response. Inferential methods based on the pertaining likelihood function are developed.

Key-words: sample selection, selection bias, Heckman model, binary variables, skew-normal distribution, count data, symmetry-modulated distributions, skew-symmetric distributions.

1 Sample selection

1.1 Nature of the problem

In observational studies, as opposed to experimental studies, a recurrent problem is the presence, at least potentially, of a sample selection mechanism, leading to a non-random sample from the target population. Although in principle the term ‘sample selection’ applies more generally, it is commonly referred to situations where the target of a study is the relationship between a response variable and a set of covariates, but individuals are observed only conditionally on the outcome of a certain selection factor, which is not independent from the variable of interest. Such dependence between the response variable and the selection factor generates a difference between the intended and the actual sampling distribution, hence an inherent bias in the inferential process.

A concrete example of this situation is discussed in the pioneering work of Heckman (1976, 1979) on the sample selection problem. In a study on the determinants of wages for female work, a linear regression model is introduced in which the wage of a worker is connected to a set of determinants, such as age, level of education, and so on. In this situation, a selection mechanism takes place because a fraction of the workers do not undertake a job whose wage is below a certain threshold; this minimal wage level, called the reservation wage, is for not fixed for all workers, but varies from subject to subject. Hence, for these subjects, we only observe the determinants of the wage, without a wage value. Clearly, plain exclusion of these cases from the analysis would lead to a bias in the coefficients of the fitted regression model, because the unobserved wages can be expected to be towards the lower end of the wage range.

The sample selection problem is widespread in all areas where observational studies are commonly in use. Social sciences in the broad sense, hence including economics, represent historically the main domain of relevance of the problem. It is then not surprising that the main body of the pertaining literature has been developed within econometrics and quantitative sociology. Notice, however, that other research domains are not excluded. For instance, the motivating example of the account of this theme by Copas & Li (1997) refers to a study of a new medical treatment where the allocation to the standard or the new type of treatment was affected by some variable not independent from the probability of success.

1.2 Heckman model

As already mentioned, fundamental work on the sample selection problem has been done by Heckman (1976, 1979), of which we now summarize the key ingredients. We phrased the exposition in a slightly different form with respect to the original, although equivalent to it, to facilitate the subsequent introduction of our construction.

Consider the case where the objective of interest is the study of the linear relationship between a response variable Y and a set of covariates x , but there is the complication that the actual observation of Y is possible when an unobserved variable U exceeds a certain threshold and the distribution of U is affected by another set of covariates w . Under assumption of joint normality of (Y, U) and linearity of the dependence of the mean values on the covariates, the probability distribution associated to the i th subject ($i = 1, \dots, n$) randomly drawn from the population is of the form

$$\begin{pmatrix} Y_i \\ U_i \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \mu_i \\ \tau_i \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix} \right), \quad \mu_i = x_i^\top \beta, \quad \tau_i = w_i^\top \gamma, \quad (1)$$

but observation of Y_i only occurs under the condition $U_i \geq 0$; the x_i vector is p -dimensional

and w_i is q -dimensional. While U_i is unobservable, what we can observe is the binary variable

$$D_i = \begin{cases} 1 & \text{if } U_i \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

so that, equivalently, observation of Y_i occurs only for cases with $D_i = 1$.

The overall available information is therefore constituted by the set of d_i binary values, the triples (y_i, x_i, w_i) for the subjects with $d_i = 1$ and by the pairs (x_i, w_i) for those with $d_i = 0$, having denoted by y_i and d_i the actual values taken on by Y_i and D_i . To compute the implied likelihood function, the ingredients are: (i) the probability of observing Y_i , namely

$$\mathbb{P}\{D_i = 1\} = \Phi(\tau_i) \quad (3)$$

where Φ denotes the $N(0, 1)$ distribution function, and (ii) the probability density function of the observed Y_i , conditionally on the event $D_i = 1$, which after some algebraic work turns out to be

$$f(y|D_i = 1) = \frac{1}{\Phi(\tau_i)\sigma} \varphi(z) \Phi\left(\frac{\tau_i + \rho z}{\sqrt{1 - \rho^2}}\right), \quad z = \frac{y - \mu_i}{\sigma}, \quad (4)$$

where $\varphi = \Phi'$. Strictly speaking, we should write $f_i(y|D_i = 1)$ in place of $f(y|D_i = 1)$ to mark its dependence on ingredients varying with the index i , but this notation would have become cumbersome if it was carried on similarly with other terms to be introduced later. The log-likelihood function is then

$$\log L = \sum_{d_i=1} \log\{\Phi(\tau_i) \times f(y_i|D_i = 1)\} + \sum_{d_i=0} \log\{1 - \Phi(\tau_i)\}. \quad (5)$$

To estimate the regression parameters β appearing in (1), the method proposed by Heckman (1976) is not directly based on this likelihood function, although an expression leading to (5) is given in his paper. In light of the limited computational resources of those years, a simpler method is presented instead, by introducing a correction factor in the regression model based on the expected value of Y_i conditionally of $U_i \geq 0$, that is, the expected value of distribution (4). After obtaining estimates of the required terms by a probit model, a second-stage least-squares estimation is then employed on the adjusted regression model; see also Heckman (1979). However, this operational simplification is not crucial; what matters more is the probability structure of the formulation.

For later reference, notice that the above formulation is built on two stochastic ingredients. We can take them to be (Y_i, U_i) or, equivalently, the 0-mean ‘error terms’ (ε_i, ζ_i) , where $\varepsilon_i = Y_i - \mu_i$ and $\zeta_i = U_i - \tau_i$, or even one of these error terms and the residual of the linear projection of the other one on the first one. Which form we consider is a matter of convenience.

Another point to annotate is that the density function (4) is of the type denoted ‘extended skew-normal’ in a stream of literature often identified by the phrase ‘skew-symmetric distributions’ or similarly ‘symmetry-modulated distributions’. A recent account of this theme is given by Azzalini & Capitanio (2014). See specifically Section 2.2 for a comprehensive treatment of the extended skew-normal distribution, including the missing algebraic details leading to (4). We shall make use of the connection with that literature to introduce our formulation later on. The connection with the skew-normal distribution has been noted by Copas & Li (1997), although they restrict it only to the case with $\tau_i = 0$.

1.3 Non-Gaussian response variables

The original Heckman construction is firmly linked to the assumption of joint normality of the (Y, U) variables. In practical work, this assumption is often made even when it is unlikely to be

appropriate, but there are cases where it would be completely untenable, at least with respect to the observable component, Y . We recall briefly a few directions of work stemming from the original Heckman construction.

An early extension of Heckman model to binary response variables has been presented by Van de Ven and Van Praag (1981). Their probability framework is similar to the normal case, but instead of Y_i we only observe its dichotomized version Y_i^* , defined similarly to (2), with Y_i replacing U_i . Hence $\mathbb{P}\{Y_i^* = 1\} = \Phi(\mu_i)$, analogously to (3). We shall return to this formulation later on.

A qualitatively different route is adopted by Terza (1998); see also Greene (2012, Section 19.5.4). While the selection mechanism is still like before, the observation Y_i is not a function of μ_i and the error term $\varepsilon_i \sim N(0, \sigma^2)$ only, like in (1), but these two ingredients determine the parameter of a distribution from which Y_i is sampled. For instance, if Y_i is taken to be of Poisson type, we could assume that $Y_i \sim \text{Poisson}(\exp(\mu_i + \varepsilon_i))$. Hence we are now considering three separate sources of variability, namely $(\varepsilon_i, \zeta_i, Y_i)$. One implication of such a scheme is that the expression of the log-likelihood function involves an additional integration over the distribution of ε_i ; see equation (3) of Terza (1998) or (19-30) of Greene (2012). Since this integration is typically not as friendly as those implicit in (3) and (4), it must be carried out numerically.

For the case of continuous response variables, a frequent criticism to Heckman's proposal is its widely recognized sensitivity to the assumption of normality. To neutralize or at least to mitigate this problem, Marchenko and Genton (2012) replace the normality assumption for (Y_i, U_i) in (1) by the one of a bivariate Student's t distribution, hence allowing for regulation of the distribution tails via the degrees of freedom. By exploiting the above-mentioned connection with results on symmetry-modulated distributions, the density in (4) is replaced by an 'extended skew- t distribution'; the factor in (3) is easy to adjust. While this construction is of adaptive type and consequently less sensitive to departure from normality than the original one of Heckman, it does not meet the formal criteria of classical robustness theory; a formulation in this framework has been developed by Zhelonkin, Genton and Ronchetti (2016).

For the analysis of count responses, Marra and Wyszynski (2016) have recently proposed a construction based on a copula function linking the response variable and the latent variable regulating selection, Y and U in our notation. The construction allows a wide choice of the copula function and of the marginal distribution of the response. In this sense, its generality is comparable with the formulation to be described in the rest of the present paper. However, it seems to us the use of a copula formulation is less natural for a discrete data context and the interpretation of the copula dependence parameter less simple compared to the continuous context, as also noted by the authors.

Within this context, the aim of the present note is the introduction of a general formulation to extend Heckman's original construction in various directions. The selection mechanism which is inherent to this situation has a natural connection with the literature on symmetry-modulated distributions, as already recalled. This connection is more on the conceptual than on the operational side, since here we move away from the requirement of symmetry on the underlying, unselected distribution of the variable of interest, which is typical of literature on symmetry-modulated distributions. However, the re-formulation of Heckman model within the conceptual framework of that literature facilitates the construction of a wider scheme, which we develop in the next section, first in general terms and then in some specific instances.

2 A broad scheme for modelling sample selection

2.1 Selective sampling as a mechanism of distribution modulation

The construction of Section 1.2 involves a bivariate random variable appearing in equation (1); denote it as (Y, U) without subscripts for notational simplicity. An equivalent stochastic representation can be obtained via the introduction of a random variable, T say, independent from the variable of interest. To be specific, if we denote

$$Z = (Y - \mu)/\sigma, \quad \alpha = \rho(1 - \rho^2)^{-1/2},$$

then

$$T = \alpha Z - (1 + \alpha^2)^{1/2} (U - \tau) \sim N(0, 1). \quad (6)$$

The pair (Y, T) is algebraically equivalent to (Y, U) with the convenient feature that $\text{cor}\{T, Y\} = \text{cor}\{T, Z\} = 0$. Note that (Y, T) is formed via the projection of the error term ζ on ε , which is one of the equivalent ways of expressing the underlying stochastic terms indicated towards the end of Section 1.2. Elementary algebra shows that the event $D = 1$ in (2) is equivalent to

$$T \leq \alpha Z + \tau (1 + \alpha^2)^{1/2}. \quad (7)$$

Hence the density of an observed y value of Y , conditionally on $D = 1$, is

$$f(y|D = 1) = \frac{1}{\Phi(\tau)} \left[\frac{1}{\sigma} \varphi(z) \Phi(\tau (1 + \alpha^2)^{1/2} + \alpha z) \right], \quad z = \frac{y - \mu}{\sigma}, \quad (8)$$

where the term inside the square brackets is the product of the marginal density of Y times the probability that $D = 1$ conditionally on $Y = y$ or, equivalently, on $Z = z$. The denominator of the leading fraction is the appropriate normalizing constant because it still holds that unconditionally $\mathbb{P}\{D = 1\} = \Phi(\tau)$; equivalently, the same fact can be show by direct integration of the term in square brackets. It is immediate that (8) coincides with (4).

If we set f to be the $N(0, \sigma^2)$ density, $G_0 = \Phi$ and $h(y) = \tau (1 + \alpha^2)^{1/2} + \alpha(y - \mu)/\sigma$, density (8) can be re-written as an instance of the more general form

$$f(y|D = 1) = \frac{1}{\pi} f(y) G_0\{h(y)\} \quad (9)$$

with normalizing constant

$$\pi = \int_{\mathbb{R}} f(y) G_0\{h(y)\} dy. \quad (10)$$

In what follows, we shall consider alternative distributions of type (9) where a ‘baseline’ density function f is modulated by a perturbation factor

$$G(y) = G_0\{h(y)\} \quad (11)$$

where G_0 is a univariate distribution function and $h(y)$ is a real-valued function. In the discrete case, f will denote a probability function and the integral in (10) must be replaced by a summation.

Denote by Y a random variable with density f and by T an independent variable with distribution function G_0 . Assume that a value y sampled from f is observed conditionally on the event $T \leq h(y)$. Then the observation of a value y generated from f takes place with conditional probability

$$\mathbb{P}\{D = 1|Y = y\} = \mathbb{P}\{T \leq h(y)|Y = y\} = G_0\{h(y)\} = G(y), \quad (12)$$

while the unconditional probability of observing a valued from f is

$$\pi = \mathbb{P}\{D = 1\} = \mathbb{E}_Y\{\mathbb{P}\{T \leq h(Y)|Y = y\}\} = \mathbb{E}_Y\{G_0\{h(Y)\}\}.$$

Since in the overwhelming majority of cases the conditional probability $G(y)$ can reasonably be assumed to be a continuous function of y , continuity is similarly assumed for G_0 and consequently for h .

It must be underlined that the adoption of the form $G_0\{h(y)\}$ for $G(y)$ in (11) does not constitute a restriction on the latter function, but only a convenient and often more meaningful way of representing the conditional probability $G(y) = \mathbb{P}\{D = 1|Y = y\}$, as seen above for the classical Heckman formulation. For any arbitrary $G(y)$, a given choice of G_0 identifies a function $h(y) = G_0^{-1}\{G(y)\}$, which is unique if G_0 is continuous. Clearly, a different choice of G_0 is linked to a different h . Which pair (G_0, h) is preferable for the given $G(y)$ is a component of the modelling process for the problem at hand; on this step the present proposal allows complete flexibility.

The connection with the literature on symmetry-modulated distributions is evident both from the expression (9), which is typical of that formulation, and from the ensuing stochastic construction via the independent variables T and Y . There are, however, also some points of distinction. One is that, as the terms itself suggests, in that literature f typically denotes the symmetric density function of a continuous random variable, possibly multivariate, and G_0 refers to symmetric univariate random variable; the rare exceptions to this setting appear in recent non-standard constructions. These symmetry conditions will not be assumed here. Another aspect, although of lesser conceptual importance, is the requirement that $h(y)$ is an odd function with respect to point of symmetry of f and G_0 . Combined with the earlier assumptions, this condition ensures that the normalizing factor (10) is $1/2$, with a major analytical simplification. This condition is not universal; for instance, it does not hold for the extended skew-normal distribution in (4) and (8). However, it applies to a large fraction of the literature of symmetry-modulated distributions, but it would be unrealistic in the present context.

An expression of type (9) can be viewed as the product of a ‘baseline’ density $f(\cdot)$, which represents the sampling distribution before censoring takes place, modulated by a perturbation factor $G(y) = G_0\{h(y)\}$ which represents the conditional probability of observing a value y generated by $f(\cdot)$. From the qualitative viewpoint, adoption of the formulation based on expression (9) has the advantage of separating, both conceptually and operationally, the choice of the uncensored distribution f and the one of the selection mechanism, expressed by the function G . Any choice of f can be combined with any choice of G .

If y_i and d_i denote the analogous quantities of those appearing in (5) with an obvious adaptation to the current formulation, in particular taking into consideration (12), the log-likelihood function takes the form

$$\log L = \sum_{d_i=1} \log [\mathbb{P}\{D_i = 1\} \times f(y_i|D_i = 1)] + \sum_{d_i=0} \log \mathbb{P}\{D_i = 0\} \quad (13)$$

$$\begin{aligned} &= \sum_{d_i=1} \log [f(y_i) \times \mathbb{P}\{D_i = 1|y_i\}] + \sum_{d_i=0} \log \mathbb{P}\{D_i = 0\} \\ &= \sum_{d_i=1} \log \{f(y_i) G(y_i)\} + \sum_{d_i=0} \log (1 - \pi_i) \end{aligned} \quad (14)$$

where π_i denotes the value of (10) evaluated for the i th individual and a similar dependence on the index i holds for other components, although not explicit in the notation, as remarked in connection with (4). Correspondingly, $\log L$ depends on parameters which appear in the ingredients f and G . As it is typical in similar cases, optimization of (14) to obtain maximum likelihood estimates (MLE) must be performed by numerical methods.

In the development below, we shall examine some specific constructions within the above scheme, where the ingredients f , G_0 , h are chosen with the aim of retaining a reasonable algebraic and numerical tractability. Hopefully, this simplicity should facilitate a meaningful interpretation from the applied viewpoint. There is no attempt, however, to present a systematic survey of the vast set of all the possible options.

2.2 Binary response variables

For expository convenience, it seems best to start from the conceptually simple case of a binary response, yet an important situation from the applied viewpoint.

Conventionally, the success and failure (uncensored) outcome on the i th subject are associated to a random variable, Y_i , taking on values 1 and 0, respectively. Typically, the probability of success is expressed as a function of covariates x_i via a form like

$$\mu_i = \mathbb{E}\{Y_i\} = \mathbb{P}\{Y_i = 1\} = P_0(x_i^\top \beta) \quad (15)$$

where P_0 is some distribution function on the real line. The more common options are the logistic and the normal distribution function, namely

$$P_0(u) = \frac{\exp(u)}{1 + \exp(u)} \quad \text{and} \quad P_0(u) = \Phi(u), \quad (16)$$

leading to the logit and the probit model for μ_i , respectively. Alternative choices for P_0 are discussed in the literature on generalized linear models (GLMs). The probability function of Y_i is then

$$f(y) = (1 - \mu_i)^{1-y} \mu_i^y, \quad y = 0, 1. \quad (17)$$

One route for modelling selective sampling is via the introduction of a bivariate normal distribution, similar to (1), followed by dichotomization of its components, leading to two correlated probit models. As mentioned earlier, this is the logic followed by Van de Ven and Van Praag (1981). To derive an inferential technique, the initial part of their exposition develops an approximate correction factor similar to the one of Heckman for normal variates, but their subsequent equation (19) presents the exact likelihood expression, which can be recognized to be analogous to our (13). In particular, the bivariate normal integrals appearing in their (19) match the joint probabilities inside the square brackets in our (13).

In this log-likelihood function, we can convert the joint probabilities in the first summation of (13) into equivalent expressions like those in (14). The term $G(y_i)$ can be expressed via the distribution function of an extended skew-normal distribution, similar to the one in (4) but with reversed role of the underlying continuous variables; an expression of the required distribution function is given in Section 2.2.3 of Azzalini and Capitanio (2014).

The resulting expression for $G(y_i)$ would be, however, quite involved. A simpler route is to write directly a model for $G(y_i)$, moving away from the assumption of an underlying bivariate normal variable. This means that we regard Y_i as a binary random variable with probability function (17), where μ_i is as in (15), and we introduce suitable ingredients $T \sim G_0$ and $h(\cdot)$ to express the conditional probability (12) of observation. In all cases, computation of π_i is elementary for binary response variables; specifically, (10) becomes

$$\pi_i = (1 - \mu_i) G(0) + \mu_i G(1).$$

In an ideal situation where subject-matter considerations in a given applied problem indicate an appropriate formulation for $G(y)$, this route should be followed. Here we discuss some

general-purpose options, driven more by considerations of simplicity, rather than linked to a particular applied problem.

A necessary requirement for $h(\cdot)$ is to incorporate the covariates w_i and the simplest way of expressing this is via τ_i , defined in (1). Formulations that arise naturally for consideration are a linear expression for $h(y)$ and $T \sim N(0, 1)$, leading to expressions such as

$$G(y) = \Phi(\tau_i + \alpha y) \quad \text{or} \quad G(y) = \Phi(\tau_i + \alpha \mu_i^{-1} y) \quad (18)$$

where $\alpha \in \mathbb{R}$ is a parameter which regulates the dependence on y and the second form introduces a form of standardization, in the sense that $\mathbb{E}\{\mu_i^{-1} Y_i\} = 1$; we shall denote $\eta_i = \alpha / \mu_i$.

However, in the present context, there is no compelling reason to stick to the assumption of normality; this is, in fact, often made for reasons like mathematical convenience or widespread familiarity rather than real belief. A mathematically simple alternative is to assume that T has a logistic distribution; this amounts to replace Φ in (18) by P_0 given in the first expression in (16). Another simple option is to say that T has an exponential variable with some fixed parameter, such as $\mathbb{E}\{T\} = 1$; we then write $T \sim \text{Expn}(1)$. In this case, to ensure that its distribution function is evaluated at positive values of the argument, we exponentiate the earlier expression of h , arriving at

$$G(y) = 1 - \exp\{-\exp(\tau_i + \alpha y)\} \quad \text{or} \quad G(y) = 1 - \exp\{-\exp(\tau_i + \eta_i y)\}, \quad (19)$$

which are related to the Gumbel distribution function.

Whatever the adopted form for $G(t)$, an ingredient of interest is a measure of association between Y and D . For a 2×2 probability table such as

$$q_{rs} = \mathbb{P}\{Y = r, D = s\}, \quad r = 0, 1, \quad s = 0, 1.$$

a classical measure of dependence is given by the log-odds ratio

$$\lambda = \log \frac{q_{00} q_{11}}{q_{10} q_{01}}.$$

A simple computation lends

$$q_{00} = \mathbb{P}\{Y = 0, D = 0\} = \mathbb{P}\{Y = 0\} \mathbb{P}\{T \geq h(0)\} = \mathbb{P}\{Y = 0\} \{1 - G(0)\}$$

and from similar computations one obtains the other probabilities, arriving at

$$\lambda = \log \frac{[1 - G(0)] G(1)}{G(0) [1 - G(1)]},$$

which, recall, depends on the index i .

2.3 Other distributions for the response variable

Among other types of data arising in applications, an important case occurs when the response variable Y represents count data. The simpler form of treatment is via the assumption of a Poisson distribution; for the i th subject we then write

$$Y_i \sim \text{Poisson}(\mu_i)$$

where μ_i denotes the mean value. The commonly used form of function relating the mean value to the covariates is

$$\mu_i = \log(x_i^\top \beta) \quad (20)$$

but also in this case others choices are possible.

As for the selection mechanism, we can still consider those introduced for binary data, such as (18) or some others mentioned in the subsequent paragraph.

The normalizing constant (10) is now represented by an infinite sum. This can be approximated by a truncated sum:

$$\sum_{k=0}^K \frac{e^{-\mu_i} \mu_i^k}{k!} G(k),$$

where the truncation point K is somewhat larger than the maximal value of y_i . A variant option is to fix a common value K across the whole set of the y_i 's.

The scheme considered so far for the binary and the Poisson distribution can be employed with some other distribution of the response variable. For instance, in cases where the Poisson distribution does not provide an adequate description of the data behaviour, a common solution is to replace it by a Negative Binomial distribution whose mean value can again be expressed as in (20) and an additional parameter regulates dispersion. For our construction, hardly anything is changed in this switch.

Another situation not feasible for the Gaussian assumption is represented by positive continuous response variables. Similarly to the framework of generalized linear models, it is then quite natural to adopt a distributional assumption such as the Exponential, the Gamma and the Inverse Gaussian family; however, this list does not intend to rule out other possibilities. Again, the modelling of the selection mechanism can be formulated via one of the expressions for $G(\cdot)$ which we have examined above. In these cases, an operational issue is whether the integral in (10) allows an explicit expression. If this is not feasible, as typically it will be the case, we can still proceed via numerical integration, at the cost of an higher computational burden.

2.4 Other forms of selection mechanism

In the earlier sections, we have discussed various choices of $G(t)$ for expressing the selection mechanism. These are by no means the only ones, however. In the case of a non-negative response variable Y , an interesting alternative is provided by the distributional assumption that $T \sim \text{Expn}(1)$ combined with the linear form

$$h(y) = \exp(\tau) + \alpha \mu^{-1} y = \lambda + \eta y,$$

say, leading to

$$G(y) = 1 - \exp\{-(\lambda + \eta y)\}. \quad (21)$$

A limitation of this choice is that we need to introduce the condition $\alpha \geq 0$ to ensure that the argument of (21) is positive. However, if such an assumption on α is plausible on the basis of subject-matter considerations, then it offers the advantage of an explicit expression for (10), in the wide range of cases where we have available a similarly explicit expression for the moment generating function of Y ; denote it by $M(\cdot)$. It is then immediate to write

$$\pi = \int_0^\infty f(y) (1 - e^{-\lambda - \eta y}) dy = 1 - e^{-\lambda} M(-\eta) \quad (22)$$

where, as usual, in the discrete case the integral sign must be interpreted as a summation.

For the distributions of Y examined above, that is, binary and Poisson, use of (22) lends

$$\pi = 1 - e^{-\lambda} [1 + \mu(e^{-\eta} - 1)] \quad \text{and} \quad \pi = 1 - \exp[-\lambda + \mu(e^{-\eta} - 1)]$$

but there are many other distributions for which $M(\cdot)$ is known in closed form, such as the Negative Binomial, Gamma, Inverse Gaussian, Binomial with arbitrary number of replicates and others more.

There are two reasons why our exposition has not focused on the form (21). One is the already-mentioned restriction that $\alpha \geq 0$, which prevents it from general usage. The other reason is that some numerical exploration has shown that the log-likelihood function (14) has, in some cases, an unpleasant behaviour. For instance, $\log L$ can be monotonic, with a maximum at $\alpha = 0$ or at $\alpha \rightarrow \infty$. However, while not appropriate for general usage, the form (21) may be suitable for specific situations.

2.5 Computational and additional inferential aspects

For the numerical maximization of the log-likelihood function, we have employed the profile log-likelihood function for α , namely

$$\log L_p(\alpha) = \log L(\alpha, \hat{\theta}(\alpha)),$$

where $\theta = (\beta^\top, \gamma^\top)^\top$ combines the two sets of parameters and $\hat{\theta}(\alpha)$ is the choice of θ which maximizes $\log L$ for a given value of α . The point $\hat{\alpha}$ which maximizes $\log L_p(\alpha)$ and the corresponding vector $\hat{\theta} = \hat{\theta}(\hat{\alpha})$ represent the MLE. In the graphical displays below, we follow the common practice of considering the so-called relative version of the log-likelihood, which in practice amounts to shift vertically $\log L_p(\alpha)$ so that its maximum value is 0.

To obtain initial values for the numerical search of θ , we fix initially $\alpha = 0$, which amounts to consider two separate generalized regression models for Y and D , free from the sample selection problem. This produces estimates of β and γ to start the subsequent overall optimization.

For any given α , the vector $\hat{\theta}(\alpha)$ is obtained by a separate numerical optimization. This can lead to a substantial computational burden if a fine grid of α values is scanned. Usually, a substantial improvement in the efficiency of the numerical search is obtained if an explicit expression of the gradient

$$\frac{d}{d\theta} \log L(\alpha, \theta) \tag{23}$$

is supplied to the optimization algorithm. General algebraic expressions for computing first and second order derivatives of the log-likelihood are given in the appendix. These need to be suitably specified for the adopted choice of f , G_0 and h .

By standard asymptotic theory, a confidence set for α can be obtained as the set of values satisfying

$$2 [\log L_p(\hat{\alpha}) - \log L_p(\alpha)] \leq q \tag{24}$$

where q denotes the quantile of the χ_1^2 distribution function at the chosen confidence level.

Standard errors for $\hat{\theta}$ can be obtained from the second-order derivatives matrix evaluated at $\hat{\alpha}$, namely

$$-\frac{d^2}{d\theta d\theta^\top} \log L(\hat{\alpha}, \theta) \Big|_{\theta=\hat{\theta}}. \tag{25}$$

When the score function is not available in an explicit form, this matrix can be obtained by numerical second order differentiation of $\log L(\hat{\alpha}, \theta)$ at $\hat{\theta}$. Since expression (25) treats α as fixed at $\hat{\alpha}$, it does not fully reflect the variability involved in the estimation process. However, this limitation affects only the one-dimensional parameter α and can reasonably assumed to be of minor importance for the assessment of standard errors of $\hat{\theta}$.

3 Numerical illustrations

3.1 German doctor visits

To illustrate the practical working of the proposed formulation, we make use of some classical datasets, repeatedly used in the specialized literature. For the case of binary response variable, we consider data presented by Riphahn, Wambach and Million (2003) from a longitudinal study concerning user preferences and usage of the German health insurance system.

We use a subset of these data to parallel the analysis presented in Example 19.13 of Greene (2012) for the binary response variable Y ‘defined to equal 1 if an individual makes at least one visit to the doctor in the survey year’, taking into account another binary variable which indicates whether the individual has subscribed a “public” health insurance. For a certain selection of covariates, the bivariate probit model of Van de Ven & Van Praag (1981) has been fitted to the data and the outcome is presented in Table 19.9 of Greene (2012).

We follow largely the same route, with some differences. One is to use the logit instead of the probit model for Y , but this is known to have little numerical effect. For the sample selection mechanism, we obviously considered the one described above. Specifically, we considered two variant forms, defining G as follows: (A) the second expression of (18), (B) the second expression of (19). Another difference is that, taking into account the longitudinal nature of the study, we only considered the first year of observation for each subject, to avoid the treatment of multiple observations taken on the same subject

Our numerical findings are summarized in Table 1 and the two variants of profile log-likelihood function are displayed in Figure 1. The most noticeable feature is the close similarity between the outcomes of the two variant forms, both in the numerical and in the graphical exhibit. Specifically, in case A, we obtained $\hat{\alpha} = -2.93$ with a 95%-level confidence interval $(-4.92, -1.70)$ using (24); in case B, $\hat{\alpha} = -3.07$ with confidence interval $(-5.40, -1.70)$. Also the values of $\hat{\theta}$ and their standard errors reported in Table 1 are very similar in the two cases.

The closeness of the two sets of results is reassuring, especially in the light of the recurrent criticism of Heckman formulation for its instability with respect to the assumption on the underlying stochastic ingredients. If one has to choose between the two models, variant A has maximized log-likelihood -6510.03 versus -6514.43 for variant B; hence A would be preferable according to Akaike and similar information criteria.

The values in Table 1 are also broadly similar to those in Table 19.9 of Greene (2012). The largest differences occurs in the two intercept terms, but these are not important for interpretation; the other terms give fairly similar indications although with some differences.

3.2 Credit cards derogatory reports

Greene (1998) examines a number of aspects in automatic credit-scoring methodology to scrutinize applications for financial credit in order to discard those which are particularly exposed to the risk of default or some other critical behaviours. In a context where a large number of such applications arise in a given time period, the adoption of an automated system is required for such scrutiny. A good example of this situation is provided by applications for credit cards, which are typically evaluated in an automated way on the basis of historical data. As the author notes, ‘In order to enter the sample used to build the model, an individual must have already been “accepted” (p. 299) with the implication that ‘a predictor of default risk in a given population of applicants can be systematically biased because it is constructed from a non-random sample of past applicants, that is, those whose applications were accepted.’ (p. 300).

Figure 1: German doctor visits data with logit model for the response variable and two choices of the selection mechanism: (A) $T \sim N(0, 1)$, $h(y) = \tau + \eta y$, (B) $T \sim \text{Expn}(1)$, $h(y) = \exp(\tau + \eta y)$.

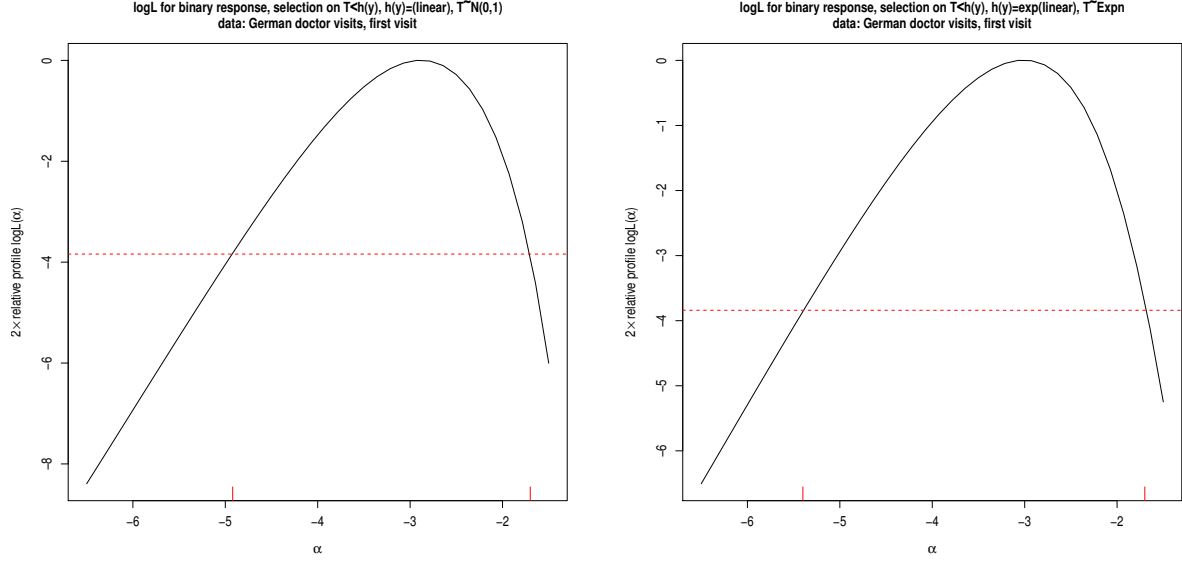


Table 1: German doctor visits data with logit model for the response variable and two choices of the selection mechanism: (A) $T \sim N(0, 1)$, $h(y) = \tau + \eta y$, (B) $T \sim \text{Expn}(1)$, $h(y) = \exp(\tau + \eta y)$.

(A) maximized $\log L = -6510.03$, $\hat{\alpha} = -2.93$ with 95%-level confidence interval $(-4.92, -1.70)$

logit model for the response variable						
	one	age	income	kids	education	married
$\hat{\beta}$	-0.49	0.0158	-0.31	-0.149	0.059	-0.045
std.err	0.15	0.0019	0.05	0.029	0.010	0.032
ratio	-3.28	8.2750	-5.79	-5.208	5.707	-1.383

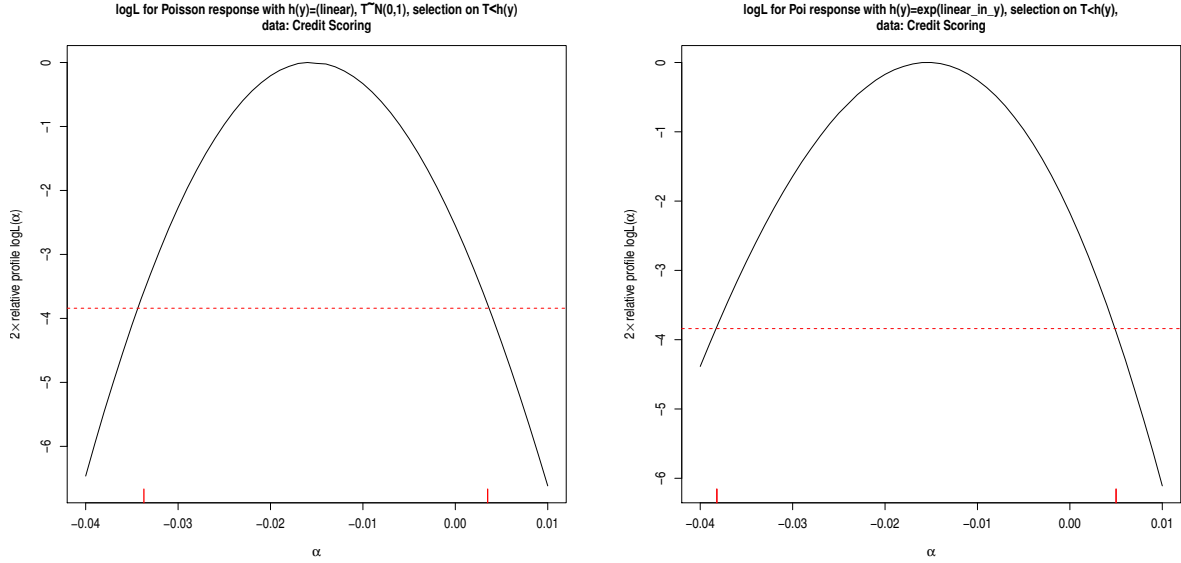
selection model				
	one	age	education	female
$\hat{\gamma}$	9.54	-0.024	-0.276	0.29
std.err	0.26	0.003	0.016	0.05
ratio	36.87	-7.205	-17.160	6.18

(B) maximized $\log L = -6514.43$, $\hat{\alpha} = -3.07$ with 95%-level confidence interval $(-5.40, -1.70)$

logit model for the response variable						
	one	age	income	kids	education	married
$\hat{\beta}$	-0.57	0.0157	-0.31	-0.109	0.064	-0.045
std.err	0.15	0.0018	0.05	0.023	0.010	0.026
ratio	-3.81	8.6514	-5.63	-4.689	6.118	-1.715

selection model				
	one	age	education	female
$\hat{\gamma}$	9.46	-0.026	-0.272	0.23
std.err	0.28	0.003	0.018	0.04
ratio	34.19	-8.031	-15.403	5.76

Figure 2: Credit cards derogatory reports with log-linear model for the mean value of the Poisson response variable and two choices of the selection mechanism: (A) $T \sim N(0, 1)$, $h(y) = \tau + \eta y$, (B) $T \sim \text{Expn}(1)$, $h(y) = \exp(\tau + \eta y)$.



Consequently, he advocates to take into consideration the sample selection mechanism, by including into consideration also subjects whose application had not been approved.

The opening sentence of Greene (1998, Section 5) is: ‘By far the most significant variable on the card-holder equation is MDRs, the number of major derogatory reports’; this is the response variable Y considered below. Greene’s treatment of the problem was based on a formulation similar to the one of Terza (1998), mentioned in Section 1.3 above, which involves the introduction of an extra latent variable ε . Another issue is that some of the covariates employed in this formulation are not included in the dataset available to us. Therefore a direct comparison with our treatment described next is not possible.

For our formulation, two choices of the selection mechanism have been considered for these data, namely the same employed in Section 3.1. Figure 2 and Table 2 provide the summary outcome of the numerical work, in the form of profile log-likelihood function, MLEs and standard errors. Also in this example the log-likelihood has a smooth nearly-quadratic behaviour for both variants of the selection model. Again, MLEs and their standard errors are in close agreement in the two variants, A and B.

4 Concluding remarks

The proposed formulation encompasses a wide range of choices for the distribution of response variable and for the sample selection mechanism. A feature which seems appealing to us is the complete separation of these two ingredients, which can be chosen independently from each other, unlike some existing proposals. Another aspect of conceptual simplicity is that our formulation involves only one latent variable in the selection mechanism, similarly to the original Heckman proposal.

Our numerical experience has indicated an appealing stability of the parameters of interest with respect to the choice of the selection mechanism. Since the range of cases considered here is limited and they are confined to discrete response variables, this point requires further

Table 2: Credit cards derogatory reports with log-linear model for the mean value of the Poisson response variable and two choices of the selection mechanism: (A) $T \sim N(0, 1)$, $h(y) = \tau + \eta y$, (B) $T \sim \text{Expn}(1)$, $h(y) = \exp(\tau + \eta y)$.

(A) maximized $\log L = -11387.63$, $\hat{\alpha} = -0.016$ with 95% confidence interval $(-0.0337, 0.0035)$

log-linear model for the response variable						
	const	Age	Income	Exp_Inc		
$\hat{\beta}$	-3.22	0.0210	0.165	1.23		
std.err	0.09	0.0023	0.016	0.16		
ratio	-35.90	9.1867	10.294	7.80		

selection model						
	Const	Age	Income	Ownrent	Adepcnt	Selfempl
$\hat{\gamma}$	0.36	-0.0014	0.217	0.224	-0.114	-0.343
std.err	0.05	0.0013	0.013	0.028	0.010	0.051
ratio	7.87	-1.0308	17.330	8.044	-11.100	-6.681

(B) maximized $\log L = -11399.83$, $\hat{\alpha} = -0.015$ with 95% confidence interval $(-0.0382, 0.005)$

log-linear model of response variable				
	const	Age	Income	Exp_Inc
$\hat{\beta}$	-3.22	0.0209	0.166	1.23
std.err	0.09	0.0023	0.016	0.16
ratio	-35.88	9.1672	10.360	7.78

selection model						
	const	Age	Income	Ownrent	Adepcnt	Selfempl
$\hat{\gamma}$	0.09	-0.0012	0.170	0.204	-0.098	-0.32
std.err	0.04	0.0012	0.010	0.024	0.009	0.05
ratio	2.29	-1.0542	17.677	8.357	-10.415	-6.67

exploration. It is conceivable that this stability is a pleasant side-effect of the discrete nature of the response variable.

We have not fully elaborated on the forms of the linear predictors for the μ_i and τ_i , for which we have retained simple parametric expressions, since our key interest was the development of the selection mechanism. It is however possible to introduce more elaborate expressions such as spline functions, following a line analogous to Marra and Wyszynski (2016).

Acknowledgments

Hyoung-Moon Kim's research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2015R1D1A1A01059161). Hea-Jung Kim's research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2015R1D1A1A01057106).

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Appendix: score function and Hessian matrix

For the overwhelming majority of cases of interest in applications, the density function f is a member of the exponential family which enter the formulation of generalized linear models; hence we focus on this situation. Following essentially the notation of McCullagh and Nelder (1989), we write the baseline density (or probability function, in the discrete case) as

$$f(y; \vartheta, \psi) = \exp \left\{ \frac{y\vartheta - b(\vartheta)}{a(\psi)} + d(y, \psi) \right\} \quad (26)$$

where $a(\cdot)$, $b(\cdot)$ and $d(\cdot)$ are known functions. In some cases, the dispersion parameters ψ is known; important instances of this type are the Poisson and the binomial distribution.

On inserting expression (26) in (14), the log-likelihood function becomes

$$\log L(\alpha, \theta, \psi) = \sum_{d_i=1} \left[\frac{y_i \vartheta_i - b(\vartheta_i)}{a_i(\psi)} + d(y_i, \psi) + \log G_0\{h(y_i)\} \right] + \sum_{d_i=0} \log(1 - \pi_i) \quad (27)$$

whose derivatives with respect to the parameters β, γ, ψ are as follows:

$$\begin{aligned} s(\beta_j) &= \frac{\partial \log L(\alpha, \theta, \psi)}{\partial \beta_j} = \sum_{d_i=1} \left[\frac{y_i - \mu_i}{V_i} + \frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \frac{\partial h(y_i)}{\partial \mu_i} \right] \frac{1}{g'(\mu_i)} x_{ij} \\ &\quad - \sum_{d_i=0} \left[\frac{\partial \pi_i / \partial \mu_i}{1 - \pi_i} \right] \frac{1}{g'(\mu_i)} x_{ij}, \quad \text{for } j = 1, \dots, p, \\ s(\gamma_h) &= \frac{\partial \log L(\alpha, \theta, \psi)}{\partial \gamma_h} = \sum_{d_i=1} \left[\frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \frac{\partial h(y_i)}{\partial \tau_i} \right] w_{ih} - \sum_{d_i=0} \left[\frac{\partial \pi_i / \partial \tau_i}{1 - \pi_i} \right] w_{ih}, \quad \text{for } h = 1, \dots, q, \\ s(\psi) &= \frac{\partial \log L(\alpha, \theta, \psi)}{\partial \psi} = \sum_{d_i=1} \left[\frac{b(\vartheta_i) - y_i \vartheta_i}{a_i^2(\psi)} a_i'(\psi) + \frac{\partial d(y_i, \psi)}{\partial \psi} \right] - \sum_{d_i=0} \frac{\partial \pi_i / \partial \psi}{1 - \pi_i} \end{aligned}$$

where $V_i = a_i(\psi) b''(\vartheta_i) = \text{var}\{Y_i\}$, $\mathbb{E}\{Y_i\} = \mu_i = b'(\vartheta_i)$, $g_0 = G'_0$ and $g(\mu_i) = x_i^\top \beta$ is called the link function.

The second order derivatives of (27) are given by the following expressions:

$$\begin{aligned} H(\beta_j, \beta_h) &= \sum_{d_i=1} \left[-\frac{1}{a(\psi)} + \left(\frac{g'_0\{h(y_i)\}}{G_0\{h(y_i)\}} - \left(\frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \right)^2 \right) \left(\frac{\partial h(y_i)}{\partial \mu_i} \right)^2 b''(\vartheta_i) \right. \\ &\quad + \frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \left(\frac{\partial^2 h(y_i)}{\partial \mu_i^2} b''(\vartheta_i) + \frac{\partial h(y_i)}{\partial \mu_i} \frac{b'''(\vartheta_i)}{b''(\vartheta_i)} \right) \\ &\quad - \left. \left\{ \frac{y_i - b'(\vartheta_i)}{V_i} + \frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \frac{\partial h(y_i)}{\partial \mu_i} \right\} \cdot \left\{ \frac{b'''(\vartheta_i)}{b''(\vartheta_i)} + \frac{b''(\vartheta_i) g''(\mu_i)}{g'(\mu_i)} \right\} \right] \frac{x_{ih} x_{ij}}{b''(\vartheta_i) (g'(\mu_i))^2} \\ &\quad + \sum_{d_i=0} \left[\frac{1}{\pi_i - 1} \left(\frac{\partial^2 \pi_i}{\partial \mu_i^2} b''(\vartheta_i) + \frac{\partial \pi_i}{\partial \mu_i} \frac{b'''(\vartheta_i)}{b''(\vartheta_i)} \right) - \frac{1}{(1 - \pi_i)^2} \left(\frac{\partial \pi_i}{\partial \mu_i} \right)^2 b''(\vartheta_i) \right. \\ &\quad + \left. \frac{\partial \pi_i / \partial \mu_i}{1 - \pi_i} \cdot \left\{ \frac{b'''(\vartheta_i)}{b''(\vartheta_i)} + \frac{b''(\vartheta_i) g''(\mu_i)}{g'(\mu_i)} \right\} \right] \frac{x_{ih} x_{ij}}{b''(\vartheta_i) (g'(\mu_i))^2}, \end{aligned}$$

$$\begin{aligned}
H(\beta_j, \gamma_h) &= \sum_{d_i=1} \left[\left\{ \frac{g'_0\{h(y_i)\}}{G_0\{h(y_i)\}} - \left(\frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \right)^2 \right\} \frac{\partial h(y_i)}{\partial \tau_i} \frac{\partial h(y_i)}{\partial \mu_i} \frac{w_{ih} x_{ij}}{g'(\mu_i)} \right. \\
&\quad \left. - \sum_{d_i=0} \left(\frac{\frac{\partial^2 \pi_i}{\partial \tau_i \partial \mu_i}}{1 - \pi_i} + \frac{\partial \pi_i}{\partial \tau_i} \frac{\partial \pi_i}{\partial \mu_i} \frac{1}{(1 - \pi_i)^2} \right) \right] \frac{w_{ih} x_{ij}}{g'(\mu_i)}, \\
H(\beta_j, \psi) &= \sum_{d_i=1} \frac{a'(\psi)(\mu_i - y_i)}{a_i^2(\psi) b''(\vartheta_i)} \frac{x_{ij}}{g'(\mu_i)} - \sum_{d_i=0} \frac{1}{(1 - \pi_i)^2} \left\{ \frac{\partial^2 \pi_i}{\partial \psi \partial \mu_i} (1 - \pi_i) + \frac{\partial \pi_i}{\partial \psi} \frac{\partial \pi_i}{\partial \mu_i} \right\} \frac{x_{ij}}{g'(\mu_i)}, \\
H(\gamma_j, \gamma_h) &= \sum_{d_i=1} \left[\left\{ \frac{g'_0\{h(y_i)\}}{G_0\{h(y_i)\}} - \left(\frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \right)^2 \right\} \left(\frac{\partial h(y_i)}{\partial \tau_i} \right)^2 + \frac{g_0\{h(y_i)\}}{G_0\{h(y_i)\}} \frac{\partial^2 h(y_i)}{\partial \tau_i^2} \right] w_{ij} w_{ih} \\
&\quad - \sum_{d_i=0} \frac{1}{(1 - \pi_i)^2} \left(\frac{\partial^2 \pi_i}{\partial \tau_i^2} (1 - \pi_i) + \left(\frac{\partial \pi_i}{\partial \tau_i} \right)^2 \right) w_{ij} w_{ih}, \\
H(\gamma_j, \psi) &= - \sum_{d_i=0} \frac{1}{(1 - \pi_i)^2} \left\{ \frac{\partial^2 \pi_i}{\partial \psi \partial \tau_i} (1 - \pi_i) + \frac{\partial \pi_i}{\partial \psi} \frac{\partial \pi_i}{\partial \tau_i} \right\} w_{ij}, \\
H(\psi, \psi) &= \sum_{d_i=1} \left\{ \frac{2(y_i \vartheta_i - b(\vartheta_i))}{a_i^3(\psi)} (a'_i(\psi))^2 - \frac{y_i \vartheta_i - b(\vartheta_i)}{a_i^2(\psi)} a''_i(\psi) + \frac{\partial^2 d(y_i, \psi)}{\partial \psi^2} \right\} \\
&\quad - \sum_{d_i=0} \frac{1}{(1 - \pi_i)^2} \left\{ \frac{\partial^2 \pi_i}{\partial \psi^2} (1 - \pi_i) + \left(\frac{\partial \pi_i}{\partial \psi} \right)^2 \right\}.
\end{aligned}$$